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Towards Fuzzy-Rough Rule Interpolation

Chengyuan Chen and Qiang Shen

***Abstract*— Fuzzy rule interpolation is an important technique for performing inferences with sparse rule bases. Even when given observations have no overlap with the antecedent values of any rule, fuzzy rule interpolation may still derive a conclusion. Nevertheless, fuzzy rule interpolation can only handle fuzziness but not roughness. Rough set theory is a useful tool to deal with incomplete knowledge, which handles roughness but not fuzziness. Fuzzy rough sets are used to extend the original concepts in rough sets. This paper proposes a novel rule interpolation method which integrates fuzzy-rough representations with rule interpolation to deal with both fuzziness and roughness. The method follows the approach of [1], [2], using transformation-based techniques to perform interpolation, and can deal with rule interpolation in a more flexible and more robust way.**

I. Introduction

The compositional rule of inference [3] was proposed as the inference mechanism to deal with fuzzy inference with dense rule bases. Given such a rule base, any input is at least partially covered by it and when an observation occurs, the conclusion can be inferred from certain rules that intersect with the observation. However, in a sparse rule base, the input may not be covered by the rule base. If a given observation has no overlap with the antecedent values of any rule, then conventional fuzzy inference methods cannot derive a conclusion because no rule can be fired. Fortunately, fuzzy rule interpolation, originally proposed in [4], [5], may still lead to certain conclusions. Yet, despite this advantage, the consequences of the method sometimes become abnormal fuzzy sets and the convexity of the derived fuzzy sets is not guaranteed [6].

In order to overcome the drawback mentioned above, significant extensions to the original fuzzy rule interpolation methods have been proposed. For instance, the slope-based technique [7] guarantees that if fuzzy sets involved in the rules and the observation are triangular, the interpolated conclusion will also be triangular. The scale and move transformation-based method [1], [2], explaining the representative values of the fuzzy sets, can handle interpolation and extrapolation for sets represented in complex polygon, Gaussian and bell-shaped fuzzy membership functions. It also guarantees the uniqueness as well as the normality and convexity of the interpolated conclusion. This method has recently been further enhanced with an adaptive mechanism such that appropriate chaining of fuzzy interpolative inferences is supported [8]. The area-based technique [9] uses the weighted average to infer the interpolated results. The cutting and transformation-based method [10] employs the cutting of geometric membership functions and the incremental and ratio transformations to support fuzzy rule interpolation.

Nevertheless, fuzzy rule interpolation can only handle fuzziness, but cannot handle roughness. In fuzzy rule bases, there may be different forms of uncertainty [11]: (1) The variables that are used in the antecedents and consequences of rules may be indiscernible. (2) The meanings of the words may be vague because words mean different things to different people. (3) An object can belong to a given degree to a set, but the degree may itself be uncertain. Much of such uncertainty in fact is considered as roughness so there will be a limitation of fuzzy rule interpolation for applications when this kind of data and knowledge is involved.

The concept of rough sets [12] was originally proposed as a mathematical tool to deal with incomplete or imperfect data and knowledge in information systems. The key notions in rough set theory are crisp equivalence classes and crisp approximations. However, rough sets can only handle roughness, but not fuzziness. Hence, fuzzy rough sets [13] have been introduced as a fuzzy generalisation of rough sets. The concept of fuzzy rough sets allows the replacement of crisp equivalence classes and crisp approximations with fuzzy equivalence classes and fuzzy approximations, respectively. So far, fuzzy rough sets have not been used to handle any combination of fuzziness and roughness in rule interpolation.

Inspired by this observation, it is potentially useful to integrate rule interpolation with fuzzy-rough concepts to deal with fuzziness and roughness conjunctively. This paper proposes an initial approach to fuzzy-rough sets-based rule interpolation. Fuzzy-rough sets, defined by the lower and upper approximate membership functions, are used to perform fuzzy-rough rule interpolation. The approach improves the flexibility of rule interpolation in dealing with inexact problem.

The rest of this paper is structured as follows. Section II reviews the relevant background of fuzzy rule interpolation, rough and fuzzy-rough sets. Section III describes the proposed fuzzy-rough rule interpolation method. Section IV gives examples to illustrate the interpolative process. Section V concludes the paper and points out important further work.

II. Background

This section briefly reviews some notions about fuzzy rule interpolation, rough and fuzzy-rough sets.

A. HS Method

The scale and move transformation-based fuzzy interpolative method is an inference mechanism following the approach of analogy (referred to as the HS method) [1], [2]. It can handle both interpolation and extrapolation. For

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simplicity, the HS method with triangular fuzzy sets is briefly reviewed in the following.

The essential notion of the representative value of a given fuzzy set is defined as the average of the x coordinates of its three key points. Given a fuzzy set A, denoted as (a_0, a_1, a_2), its representative value is:

$$\text{Rep}(A) = \frac{a_0 + a_1 + a_2}{3}$$

Suppose two adjacent rules $A_1 \Rightarrow B_1$, $A_2 \Rightarrow B_2$ and the observation A^*, which is located between fuzzy sets A_1 and A_2, are given. $A_i = (a_{i0}, a_{i1}, a_{i2})$, $B_i = (b_{i0}, b_{i1}, b_{i2})$, $i = 1, 2$, and $A^* = (a_0, a_1, a_2)$. Also, denote the outcome by $B^* = (b_0, b_1, b_2)$.

The simplest interpolation which is linear can be written as:

$$\frac{d(A^*, A_1)}{d(A^*, A_2)} = \frac{d(B^*, B_1)}{d(B^*, B_2)},$$

where $d(.,.)$ is typically the Euclidean distance (though other distance metrics may be used as alternatives for this).

The first step is to generate a new fuzzy set A' using A_1 and A_2, which has the same representative value as A^*. For this, the following is created first:

$$\lambda_{\text{Rep}} = \frac{d(A_1, A^*)}{d(A_1, A_2)} = \frac{d(\text{Rep}(A_1), \text{Rep}(A^*))}{d(\text{Rep}(A_1), \text{Rep}(A_2))},$$

where $d(A_1, A_2) = d(\text{Rep}(A_1), \text{Rep}(A_2))$ represents the distance between A_1 and A_2.

From this, $a_0^{'}$, $a_1^{'}$ and $a_2^{'}$ of $A^{'}$ are calculated as follows:

$$\begin{aligned}
a_0^{'} &= (1 - \lambda_{\text{Rep}}) \times a_{10} + \lambda_{\text{Rep}} \times a_{20},\\
a_1^{'} &= (1 - \lambda_{\text{Rep}}) \times a_{11} + \lambda_{\text{Rep}} \times a_{21},\\
a_2^{'} &= (1 - \lambda_{\text{Rep}}) \times a_{12} + \lambda_{\text{Rep}} \times a_{22},
\end{aligned}$$

which are collectively abbreviated to

$$A^{'} = (1 - \lambda_{\text{Rep}}) \times A_1 + \lambda_{\text{Rep}} \times A_2$$

The second step is to generate the consequent fuzzy set $B^{'}$ in a similar way to the first, $B^{'}$ can be obtained as follows:

$$\begin{aligned}
b_0^{'} &= (1 - \lambda_{\text{Rep}}) \times b_{10} + \lambda_{\text{Rep}} \times b_{20},\\
b_1^{'} &= (1 - \lambda_{\text{Rep}}) \times b_{11} + \lambda_{\text{Rep}} \times b_{21},\\
b_2^{'} &= (1 - \lambda_{\text{Rep}}) \times b_{12} + \lambda_{\text{Rep}} \times b_{22},
\end{aligned}$$

with abbreviated notation

$$B^{'} = (1 - \lambda_{\text{Rep}}) \times B_1 + \lambda_{\text{Rep}} \times B_2$$

As a result, $A^{'} \Rightarrow B^{'}$ is derived from $A_1 \Rightarrow B_1$ and $A_2 \Rightarrow B_2$. Suppose that a certain degree of similarity between $A^{'}$ and A^* is established, it is intuitive to require that the consequent parts $B^{'}$ and B^* attain the same similarity degree. The HS method uses the following two transformations to ensure this.

Scale Transformation: Given a scale rate s ($s \geq 0$), transform the current support $(a_2 - a_0)$ into a new support $(s * (a_2 - a_0))$ while keeping the same representative value and the ratio of left-support $(a_1^{'} - a_0^{'})$ to right-support $(a_2^{'} - a_1^{'})$ as those of its original, i.e., $\text{Rep}(A^{'}) = \text{Rep}(A)$ and $((a_1^{'} - a_0^{'})/(a_2^{'} - a_1^{'})) = ((a_1 - a_0)/(a_2 - a_1))$.

From this, $a_0^{'}$, $a_1^{'}$ and $a_2^{'}$ of $A^{'}$ are calculated as follows:

$$\begin{aligned}
a_0^{'} &= \frac{a_0(1 + 2s) + a_1(1 - s) + a_2(1 - s)}{3},\\
a_1^{'} &= \frac{a_0(1 - s) + a_1(1 + 2s) + a_2(1 - s)}{3},\\
a_2^{'} &= \frac{a_0(1 - s) + a_1(1 - s) + a_2(1 + 2s)}{3}
\end{aligned}$$

Move Transformation: Given a moving distance l, transform the current support $(a_2 - a_0)$ from the starting position a_0 to a new starting position $a_0 + l$ while keeping the same representative value and length of support of the transformed fuzzy set as its original, i.e., $\text{Rep}(A^{'}) = \text{Rep}(A)$ and $a_2^{'} - a_0^{'} = a_2 - a_0$.

From this, $a_0^{'}$, $a_1^{'}$ and $a_2^{'}$ of $A^{'}$ are calculated as follows:

$$\begin{aligned}
a_0^{'} &= a_0 + l,\\
a_1^{'} &= a_1 - 2l,\\
a_2^{'} &= a_2 + l
\end{aligned}$$

The third step is to calculate the similarity degree in terms of scale rate s and moving distance l between $A^{'}$ and A^*, and then obtain the interpolated conclusion B^* by transforming $B^{'}$ with the same scale rate and moving distance. For more details, please refer to [1].

B. Rough and Fuzzy-Rough Sets

Central to rough sets is the concept of indiscernibility. Let $I = (U, A)$ be an information system, where U is a nonempty set (the universe) of finite objects and A is a nonempty finite set of attributes such that $a : U \to V_a$ for every $a \in A$. With any $P \subseteq A$ there is a crisp equivalence relation $IND(P)$ [14]:

$$IND(P) = \{(x, y) \in U^2 \mid \forall a \in P,\ a(x) = a(y)\}$$

Let $X \subseteq U$, X can be approximated using the information contained within P by constructing the crisp P-lower and P-upper approximations of the crisp set X:

$$\begin{aligned}
\underline{P}X &= \{x \mid [x]_P \subseteq X\},\\
\overline{P}X &= \{x \mid [x]_P \cap X \neq \emptyset\}.
\end{aligned}$$

The tuple $< \underline{P}X, \overline{P}X >$ is called a rough set.

In the same way that crisp equivalence classes and crisp approximations are central to rough sets, fuzzy equivalence classes and fuzzy approximations are central to fuzzy-rough sets [15].

1) Fuzzy Equivalence Classes: The concept of crisp equivalence classes can be extended by the inclusion of a fuzzy similarity relation S on the universe, which determines the extent to which two elements are similar in S. The usual properties, reflexivity ($\mu_S(x, x) = 1$), symmetry ($\mu_S(x, y) = \mu_S(y, x)$), and transitivity ($\mu_S(x, z) \geq \mu_S(x, y) \wedge \mu_S(y, z)$, where $\wedge$ is a t-norm) hold [14].

Using the fuzzy similarity relation, the fuzzy equivalence class $[x]_S$ for objects close to x can be defined by:

$$\mu_{[x]_S}(y) = \mu_S(x, y) \tag{1}$$

2) Fuzzy Approximations: Fuzzy P-lower and P-upper approximations were originally given as follows [16]:

$$\begin{aligned}\mu_{\underline{P}X}(F_i) &= \inf_x \max\{1-\mu_{F_i}(x), \mu_X(x)\} \ \forall i,\\ \mu_{\overline{P}X}(F_i) &= \sup_x \min\{\mu_{F_i}(x), \mu_X(x)\} \ \forall i,\end{aligned} \tag{2}$$

where F_i is a fuzzy equivalence class belonging to U/P, and X is the (fuzzy) concept to be approximated. The tuple $< \underline{P}X, \overline{P}X >$ is called a fuzzy rough set. It can be seen that these definitions degenerate to conventional rough sets when all equivalence classes are crisp.

Let A and R be a fuzzy set and an equivalence relation over U, respectively, where U is the universe of discourse. Then, a similar expression of Equation (2) is:

$$\begin{aligned}\mu_{\underline{R}(A)}([x]_R) &= \inf_{y\in U} \max\{1-\mu_{[x]_R}(y), \mu_A(y)\},\\ \mu_{\overline{R}(A)}([x]_R) &= \sup_{y\in U} \min\{\mu_{[x]_R}(y), \mu_A(y)\}.\end{aligned} \tag{3}$$

Let $U = [0,1]$, based on Equation (1), Equation (3) becomes:

$$\begin{aligned}\mu_{\underline{R}(A)}(x) &= \inf_{y\in U} \max\{1-\mu_R(x,y), \mu_A(y)\},\\ \mu_{\overline{R}(A)}(x) &= \sup_{y\in U} \min\{\mu_R(x,y), \mu_A(y)\},\end{aligned}$$

which are termed the fuzzy lower and upper approximations of fuzzy set A, and the tuple $< \underline{R}(A), \overline{R}(A) >$ is termed a fuzzy-rough set. For simplicity, triangular membership functions are considered to demonstrate the basic ideas of the present work below.

III. Fuzzy-Rough-Based Rule Interpolation

Definition 3.1: A fuzzy-rough set A is defined by the lower approximate membership function $\underline{A}$ and the upper approximate membership function $\overline{A}$, i.e., $A =< \underline{A}, \overline{A} >$, as shown in Figure 1, where $\underline{A} = (\underline{a_0}, \underline{a_1}, \underline{a_2}; \text{Hgt}\{\underline{A}\})$ and $\overline{A} = (\overline{a_0}, \overline{a_1}, \overline{a_2}; \text{Hgt}\{\overline{A}\})$, $\underline{a_0}$, $\underline{a_1}$, $\underline{a_2}$, and $\overline{a_0}$, $\overline{a_1}$, $\overline{a_2}$ denote the three key points: the left and right extreme points and the highest points of x coordinates, $\text{Hgt}\{\underline{A}\}$ and $\text{Hgt}\{\overline{A}\}$ denote the maximum membership values of $\underline{A}$ and $\overline{A}$, respectively, and $\overline{a_0} \le \underline{a_0}$, $\underline{a_2} \le \overline{a_2}$, $0 \le \text{Hgt}\{\underline{A}\} \le \text{Hgt}\{\overline{A}\} \le 1$.

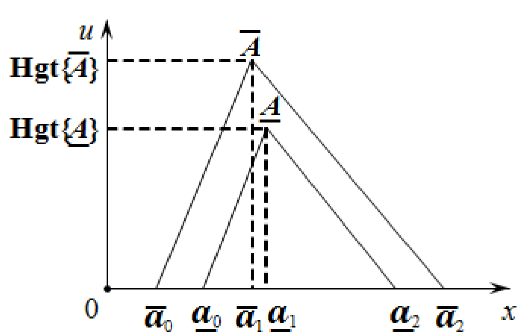

Fig. 1. Lower approximate membership function $\underline{A}$ and upper approximate membership function $\overline{A}$ of a triangular fuzzy-rough set A

The lower and upper approximate membership functions $\underline{A}$ and $\overline{A}$ denote the degree of roughness of the fuzzy-rough set A. The closer the shapes of $\underline{A}$ and $\overline{A}$, the lower the roughness of A. When $\underline{A}$ coincides with $\overline{A}$, the fuzzy-rough set degenerates to a conventional fuzzy set.

Definition 3.2: Given a fuzzy-rough set A as defined in Figure 1, with the six distinct coordinates of the fuzzy-rough set being $(\underline{a_0}, 0)$, $(\underline{a_1}, \text{Hgt}\{\underline{A}\})$, $(\underline{a_2}, 0)$, $(\overline{a_0}, 0)$, $(\overline{a_1}, \text{Hgt}\{\overline{A}\})$ and $(\overline{a_2}, 0)$, the lower representative value $\text{Rep}(\underline{A})$ and the upper representative value $\text{Rep}(\overline{A})$ of the fuzzy-rough set A can be defined by [1]

$$\begin{aligned}\text{Rep}(\underline{A})_x &= \frac{\underline{a_0}+\underline{a_1}+\underline{a_2}}{3},\\ \text{Rep}(\underline{A})_y &= \frac{0+\text{Hgt}\{\underline{A}\}+0}{3},\\ \text{Rep}(\overline{A})_x &= \frac{\overline{a_0}+\overline{a_1}+\overline{a_2}}{3},\\ \text{Rep}(\overline{A})_y &= \frac{0+\text{Hgt}\{\overline{A}\}+0}{3},\end{aligned} \tag{4}$$

where x and y denote the x coordinate and the y coordinate, respectively.

Definition 3.3: The lower standard deviation $\text{Std}(\underline{A})$ and the upper standard deviation $\text{Std}(\overline{A})$ are defined as follows:

$$\begin{aligned}\text{Std}(\underline{A}) &= \sqrt{\frac{\sum_{i=0}^{2}(\underline{a_i}-\text{Rep}(\underline{A})_x)^2}{3}},\\ \text{Std}(\overline{A}) &= \sqrt{\frac{\sum_{i=0}^{2}(\overline{a_i}-\text{Rep}(\overline{A})_x)^2}{3}}\end{aligned} \tag{5}$$

A small standard deviation value implies that the elements of the attribute tend to be close to the lower (upper) representative value. That is, the smaller the standard deviation, the smaller the area of the lower (upper) approximate membership function.

In order to obtain a unique value to act as the overall representative value of a given fuzzy-rough set, the concept of weighted values of the lower and upper approximate membership functions will be defined first.

Definition 3.4: The lower weighted value $\underline{W_A}$ and the upper weighted value $\overline{W_A}$ are defined as follows:

$$\begin{aligned}\underline{W_A} &= \frac{\text{Rep}(\underline{A})_y}{\text{Rep}(\underline{A})_y+\text{Rep}(\overline{A})_y},\\ \overline{W_A} &= \frac{\text{Rep}(\overline{A})_y}{\text{Rep}(\underline{A})_y+\text{Rep}(\overline{A})_y},\end{aligned} \tag{6}$$

where $\underline{W_A}+\overline{W_A}=1$.

Definition 3.5: Given a fuzzy-rough set A, the representative value $\text{Rep}(A)$ of A is calculated as follows:

$$\begin{aligned}\text{Rep}(A) =& \underline{W_A}\times(\text{Rep}(\underline{A})_x+\text{Rep}(\underline{A})_y-\text{Std}(\underline{A}))\\ &+\overline{W_A}\times(\text{Rep}(\overline{A})_x+\text{Rep}(\overline{A})_y-\text{Std}(\overline{A}))\end{aligned} \tag{7}$$

Note that in the above definition, the lower and upper standard deviations are deducted from the lower and upper representative values. This is necessary because otherwise, the same representative value would be derived from different shapes of fuzzy-rough sets A and A' if $\text{Rep}(\underline{A})_x =$

$\text{Rep}(\underline{A}')_x$ and $\text{Rep}(\underline{A})_y = \text{Rep}(\underline{A}')_y$ (see Examples 4.2 and 4.3).

Definition 3.6: The proportional value λ_{Rep} of the fuzzy-rough sets A_1, A and A_2 is calculated as follows [1]:

$$\lambda_{\text{Rep}} = \frac{\text{Rep}(A) - \text{Rep}(A_1)}{\text{Rep}(A_2) - \text{Rep}(A_1)}, \tag{8}$$

where $\text{Rep}(A_2) - \text{Rep}(A_1) \neq 0$.

Note that the above condition always holds. Otherwise, rules R_1 and R_2 are at least overlapping if not identical, which would make no sense for the need of interpolation.

Suppose that there are a fuzzy-rough rule base and a fuzzy-rough observation, the inference model for fuzzy-rough rule interpolation is as follows:

$$\begin{aligned}
&\text{Rule 1}: \ \text{If } X_1 \text{ is } A_1 \text{ then } Y_1 \text{ is } B_1\\
&\text{Rule 2}: \ \text{If } X_2 \text{ is } A_2 \text{ then } Y_2 \text{ is } B_2\\
&\text{Observation}: \ X \text{ is } A\\
&\text{Conclusion}: \ Y \text{ is } B
\end{aligned}$$

where A_1, A_2, A, B_1 and B_2 are fuzzy-rough sets, $A_1 \Rightarrow B_1$ and $A_2 \Rightarrow B_2$ are two adjacent and disjoint fuzzy-rough rules, as shown in Figure 2. This follows exactly from the conventional fuzzy interpolative techniques such as those repeated in [1], [2].

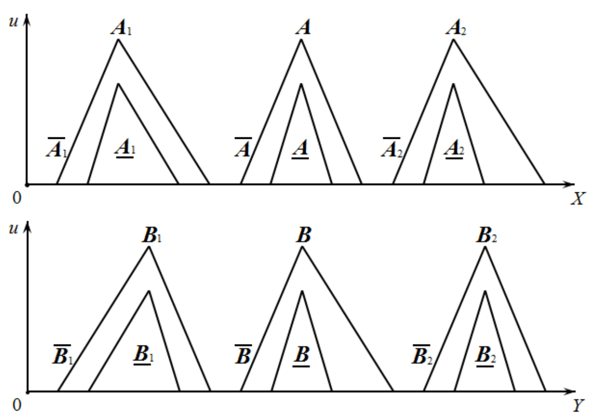

Fig. 2. Fuzzy-rough rule interpolation with triangular membership functions

The algorithm of the proposed method is presented in Algorithm 1, where the interpolated conclusion can be derived in three steps which are explained below.

Step 1: Based on Equation (4), the lower and upper representative values $\text{Rep}(\underline{A})_k$ and $\text{Rep}(\overline{A})_k$ are calculated to approximate the lower and upper approximate membership functions of the fuzzy-rough set A, respectively, $k \in \{x, y\}$. Based on Equation (5), the lower and upper standard deviations are computed to approximate the standard deviations of the lower and upper approximate membership functions, respectively. Based on Equation (6), the lower and upper weighted values $\underline{W_A}$ and $\overline{W_A}$ are calculated to obtain the weighted values of the maximum membership values $\text{Hgt}\{\underline{A}\}$ and $\text{Hgt}\{\overline{A}\}$, respectively.

Step 2: Based on Equation (7), the representative values $\text{Rep}(A_1)$, $\text{Rep}(A)$ and $\text{Rep}(A_2)$ are calculated using the

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Algorithm 1 Fuzzy-rough-based rule interpolation algorithm

Initialize: A fuzzy-rough set $A = < \underline{A}, \overline{A} > = < (\underline{a_0}, \underline{a_1}, \underline{a_2}; \text{Hgt}\{\underline{A}\}), (\overline{a_0}, \overline{a_1}, \overline{a_2}; \text{Hgt}\{\overline{A}\}) >$

Calculating $\text{Rep}(A)$:

Input: A
$\text{Rep}(\underline{A})_x$, $\text{Rep}(\overline{A})_x \leftarrow \underline{a_i}$, $\overline{a_i}$, $0 \leq i \leq 2$
$\text{Rep}(\underline{A})_y$, $\text{Rep}(\overline{A})_y \leftarrow \text{Hgt}\{\underline{A}\}$, $\text{Hgt}\{\overline{A}\}$
$\text{Std}(\underline{A})$, $\text{Std}(\overline{A}) \leftarrow \text{Rep}(\underline{A})_x$, $\text{Rep}(\overline{A})_x$, $\underline{a_i}$, $\overline{a_i}$, $0 \leq i \leq 2$
$\underline{W_A}$, $\overline{W_A} \leftarrow \text{Rep}(\underline{A})_y$, $\text{Rep}(\overline{A})_y$
$\text{Rep}(A) = \underline{W_A} \times (\text{Rep}(\underline{A})_x + \text{Rep}(\underline{A})_y - \text{Std}(\underline{A}))$
$\quad + \overline{W_A} \times (\text{Rep}(\overline{A})_x + \text{Rep}(\overline{A})_y - \text{Std}(\overline{A}))$
return $\text{Rep}(A)$

Calculating B:

Input: $\text{Rep}(A_1)$, $\text{Rep}(A)$, $\text{Rep}(A_2)$, B_1 and B_2
$\lambda_{\text{Rep}} \leftarrow (\text{Rep}(A) - \text{Rep}(A_1)) : (\text{Rep}(A_2) - \text{Rep}(A_1))$
for $i = 0$ to 2 **do**
$\quad \underline{b_i} = (1 - \lambda_{\text{Rep}}) \times \underline{b_{1i}} + \lambda_{\text{Rep}} \times \underline{b_{2i}}$
$\quad \overline{b_i} = (1 - \lambda_{\text{Rep}}) \times \overline{b_{1i}} + \lambda_{\text{Rep}} \times \overline{b_{2i}}$
end for
$\text{Hgt}\{\underline{B}\} = (1 - \lambda_{\text{Rep}}) \times \text{Hgt}\{\underline{B_1}\} + \lambda_{\text{Rep}} \times \text{Hgt}\{\underline{B_2}\}$
$\text{Hgt}\{\overline{B}\} = (1 - \lambda_{\text{Rep}}) \times \text{Hgt}\{\overline{B_1}\} + \lambda_{\text{Rep}} \times \text{Hgt}\{\overline{B_2}\}$
return B

---

results of Step 1. Based on Equation (8), the proportional value λ_{Rep} is calculated by these three representative values.

Step 3: The interpolated conclusion is derived from the fuzzy-rough rules and the observation using fuzzy-rough rule interpolation. Finally, $B = < \underline{B}, \overline{B} > = < (\underline{b_0}, \underline{b_1}, \underline{b_2}; \text{Hgt}\{\underline{B}\}), (\overline{b_0}, \overline{b_1}, \overline{b_2}; \text{Hgt}\{\overline{B}\}) >$ is obtained by

$$\begin{aligned}
&\underline{b_i} = (1 - \lambda_{\text{Rep}}) \times \underline{b_{1i}} + \lambda_{\text{Rep}} \times \underline{b_{2i}},\\
&\overline{b_i} = (1 - \lambda_{\text{Rep}}) \times \overline{b_{1i}} + \lambda_{\text{Rep}} \times \overline{b_{2i}},\\
&\text{Hgt}\{\underline{B}\} = (1 - \lambda_{\text{Rep}}) \times \text{Hgt}\{\underline{B_1}\} + \lambda_{\text{Rep}} \times \text{Hgt}\{\underline{B_2}\},\\
&\text{Hgt}\{\overline{B}\} = (1 - \lambda_{\text{Rep}}) \times \text{Hgt}\{\overline{B_1}\} + \lambda_{\text{Rep}} \times \text{Hgt}\{\overline{B_2}\},
\end{aligned} \tag{9}$$

where $0 \leq i \leq 2$.

IV. Experimentation and Discussions

This section employs several examples to illustrate the proposed fuzzy-rough rule interpolation method.

Example 4.1: This case considers the proposed method involving only triangular fuzzy-rough sets. Let A_1, A, A_2, B_1 and B_2 be fuzzy-rough sets, where

$$\begin{aligned}
A_1 &= < (1, 4, 5; 0.7), (0, 5, 6; 1) >\\
A &= < (7.5, 8, 9; 0.7), (7, 8, 10; 1) >\\
A_2 &= < (12, 13, 13.5; 0.7), (11, 13, 14; 1) >\\
B_1 &= < (1.5, 2, 3; 0.5), (0, 2, 4; 1) >\\
B_2 &= < (11, 11.5, 12; 0.5), (10, 11, 13; 1) >
\end{aligned}$$

First, the lower and upper representative values, standard deviations and weighted values are calculated according to Equations (4), (5) and (6). Then, the representative values

$\text{Rep}(A_1) = 1.578$, $\text{Rep}(A) = 7.566$, $\text{Rep}(A_2) = 12.037$ and the proportional value $\lambda_{\text{Rep}} = 0.573$ are calculated from Equations (7) and (8). Finally, the interpolated conclusion $B =< (6.94, 7.44, 8.15; 0.5), (5.73, 7.15, 9.15; 1) >$ is obtained by Equation (9), as shown in Figure 3. Clearly, the result reflects the intuition for the need of fuzzy-rough rule interpolation very well.

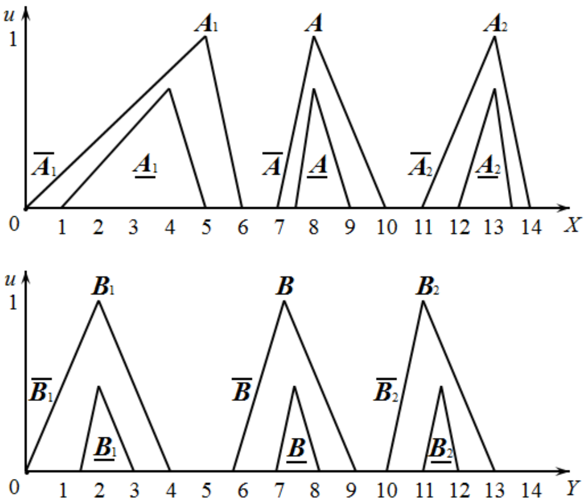

Fig. 3. Fuzzy-rough rule interpolation of Example 4.1

Example 4.2: This case considers the proposed method involving singleton-valued conditions. Let A_1, A, A_2, B_1 and B_2 be fuzzy-rough sets, where

$$A_1 =< (3, 3, 3; 1), (3, 3, 3; 1) >$$
$$A =< (6, 7, 8; 0.6), (5, 7, 9; 1) >$$
$$A_2 =< (12, 12, 12; 1), (12, 12, 12; 1) >$$
$$B_1 =< (4, 4, 4; 1), (4, 4, 4; 1) >$$
$$B_2 =< (10.5, 11.5, 12; 0.5), (10, 11.5, 13; 1) >$$

First, the lower and upper representative values, standard deviations and weighted values are calculated according to Equations (4), (5) and (6). Then, the representative values $\text{Rep}(A_1) = 3.333$, $\text{Rep}(A) = 5.957$, $\text{Rep}(A_2) = 12.333$ and the proportional value $\lambda_{\text{Rep}} = 0.291$ are calculated from Equations (7) and (8). Finally, the interpolated conclusion $B =< (5.89, 6.19, 6.33; 0.85), (5.75, 6.19, 6.62; 1) >$ is obtained by Equation (9), as shown in Figure 4. Again, the result is of very good intuitive appeal.

Example 4.3: This case considers a similar situation to Example 4.2, but the shape of the observation is different. $\text{Rep}(\underline{A})_x + \text{Rep}(\underline{A})_y$ is of the same value in these two cases. It can be seen that the interpolated results are of different values owing to the contribution of the standard deviations.

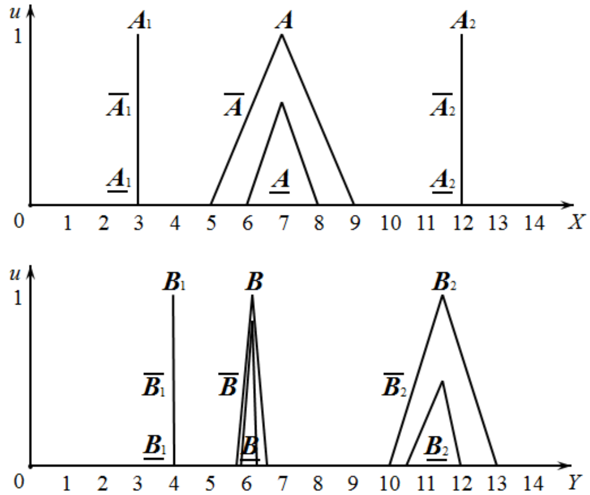

Fig. 4. Fuzzy-rough rule interpolation of Example 4.2

Let A_1, A, A_2, B_1 and B_2 be fuzzy-rough sets, where

$$A_1 =< (3, 3, 3; 1), (3, 3, 3; 1) >$$
$$A =< (6.5, 7, 7.5; 0.6), (5, 7, 9; 1) >$$
$$A_2 =< (12, 12, 12; 1), (12, 12, 12; 1) >$$
$$B_1 =< (4, 4, 4; 1), (4, 4, 4; 1) >$$
$$B_2 =< (10.5, 11.5, 12; 0.5), (10, 11.5, 13; 1) >$$

First, the lower and upper representative values, standard deviations and weighted values are calculated according to Equations (4), (5) and (6). Then, the representative values $\text{Rep}(A_1) = 3.333$, $\text{Rep}(A) = 6.11$, $\text{Rep}(A_2) = 12.333$ and the proportional value $\lambda_{\text{Rep}} = 0.308$ are calculated from Equations (7) and (8). Finally, the interpolated conclusion $B =< (6.01, 6.31, 6.47; 0.85), (5.85, 6.31, 6.78; 1) >$ is obtained by Equation (9), as shown in Figure 5.

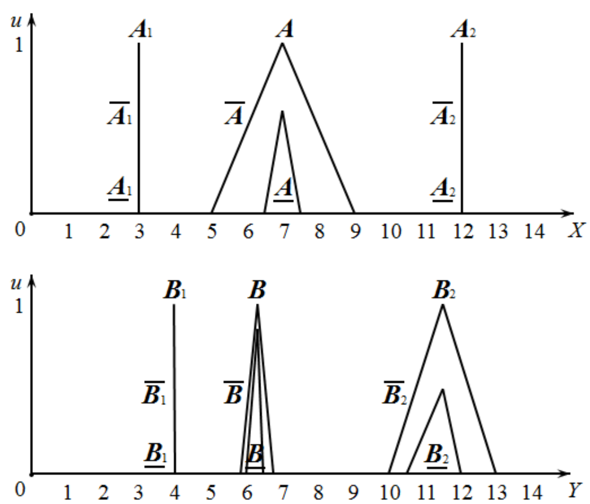

Fig. 5. Fuzzy-rough rule interpolation of Example 4.3

From the last two examples, it follows that if certain components involved in the given rules are singleton-valued, the interpolated conclusion remains a fuzzy-rough set.

Example 4.4: This case considers a special condition that all the fuzzy-rough sets degenerate to the conventional fuzzy sets, i.e., $\underline{A_i} = \overline{A_i}$ and $\underline{B_i} = \overline{B_i}$. Let A_1, A, A_2, B_1 and B_2 be fuzzy-rough sets, where

$$A_1 =< (0,5,6;1),(0,5,6;1) >$$
$$A =< (7,8,9;1),(7,8,9;1) >$$
$$A_2 =< (11,13,14;1),(11,13,14;1) >$$
$$B_1 =< (0,2,3;1),(0,2,3;1) >$$
$$B_2 =< (10,11,12;1),(10,11,12;1) >$$

First, the lower and upper representative values, standard deviations and weighted values are calculated according to Equations (4), (5) and (6). Then, the representative values $\text{Rep}(A_1) = 1.375$, $\text{Rep}(A) = 7.517$, $\text{Rep}(A_2) = 11.753$ and the proportional value $\lambda_{\text{Rep}} = 0.592$ are calculated from Equations (7) and (8). Finally, the interpolated conclusion $B =< (5.92, 7.33, 8.33; 1), (5.92, 7.33, 8.33; 1) >$ is obtained by Equation (9), as shown in Figure 6.

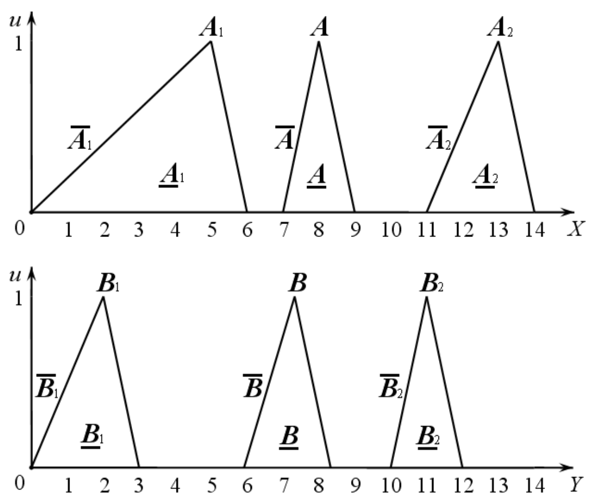

Fig. 6. Fuzzy-rough rule interpolation of Example 4.4

From this example, it follows that if everything is a conventional fuzzy set, i.e., no roughness involved, the interpolated conclusion is the same as the existing conventional fuzzy rule interpolation. Putting all four examples together, this empirical investigation demonstrates that the proposed fuzzy-rough rule interpolation method is a useful method to deal with both fuzziness and roughness in rule interpolation.

V. Conclusions

This paper has proposed an initial idea for the development of fuzzy-rough rule interpolation. It has introduced the concepts of lower and upper approximate membership functions and presented a preliminary algorithm for fuzzy-rough rule interpolation, assuming that rules involving fuzzy-rough-valued attributes are available. The algorithm works by first using the lower and upper representative values to compute the lower and upper approximate membership functions of fuzzy-rough sets, and then deriving the interpolated conclusion using the proportional value which is calculated by the representative values. The proposed approach can deal with rule interpolation in a more flexible and more robust way than conventional fuzzy rule interpolation.

The present work only uses triangular fuzzy-rough sets. However, the underlying idea seems to be more general, but this needs verification by extending the current method to coping with other types of fuzzy-rough set (e.g., trapezoidal and polygonal). Also, only rules containing single antecedent and single conclusion are considered in this paper. It would be very interesting to investigate how this may be extended to multiple antecedents situations. Whilst empirical results have shown that this approach reflects well the intuition of using fuzzy-rough sets to address both fuzziness and roughness at the same time, theoretical proof in terms of it being a generalisation of the conventional fuzzy rule interpolation method, represented by [1], remains as active research.

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